Epicyclic orbital oscillations in Newton's and Einstein's gravity from the geodesic deviation equation

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Abstract

In a recent paper Abramowicz and Kluźniak [1] have discussed the problem of epicyclic oscillations in Newton's and Einstein's dynamics and have shown that Newton's dynamics in a properly curved three-dimensional space is identical to test-body dynamics in the three-dimensional optical geometry of Schwarzschild space-time. One of the main results of this paper was the proof that different behaviour of radial epicyclic frequency and Keplerian frequency in Newtonian and General Relativistic regimes had purely geometric origin contrary to claims that nonlinearity of Einstein's theory was responsible for this effect.

In this paper we obtain the same result from another perspective: by representing these two distinct problems (Newtonian and Einstein's test body motion in central gravitational field) in a uniform way — as a geodesic motion. The solution of geodesic deviation equation reproduces the well known results concerning epicyclic frequencies and clearly demonstrates geometric origin of the difference between Newtonian and Einstein's problems.

Key Words: Geodesic deviation, Jacobi geometry, epicyclic oscillations, Schwarzschild spacetime

1 Introduction

In a recent paper Abramowicz and Kluźniak [1] have discussed the problem of epicyclic oscillations in Newton's and Einstein's dynamics and have shown that Newton's dynamics in a properly curved three-dimensional space is identical to test-body dynamics in the three-dimensional optical geometry of Schwarzschild space-time. Their discussion was motivated by the theory of accretion disks around black holes and neutron stars which is based on assumption that accreting matter moves on nearly circular geodesic trajectories. One of the strong field effects that should be present in this context, as pointed out by Abramowicz and Kluźniak in a recent series of papers [2], is the possibility of parametric resonance (preferably 3:2) between vertical and radial epicyclic frequencies of perturbed circular orbits. It has been conjectured [2] that this effect is indeed responsible for the observed double peaked QPOs [3].

It is quite well known that in the Schwarzschild (or Kerr) spacetime radial epicyclic frequency ω_r is lower than orbital frequency ω_K (and vanishes at the marginally stable orbit) unlike in the Newtonian gravity where these two frequencies are equal. As recalled by the authors of [1] many people attributed this different behaviour to the nonlinearity of the Einstein's theory of gravity. Therefore one of principal motivations for [1] (besides making a brilliant use of the so called optical geometry [4]) was to demonstrate the purely geometric origin of this effect. In order to achieve this Abramowicz and Kluźniak have represented the Einstein equations (in optical geometry) for the motion on a circular orbit in Schwarzschild space-time in the form of Newton's equations in certain curved 3-dimensional space. Then they were able to calculate the epicyclic frequencies in a uniform way (i.e. from the same equation) and show explicitly that the aforementioned difference ($\omega_r < \omega_K$ in Einstein's gravity vs. $\omega_r = \omega_K$ in Newton's gravity) has purely geometric origin.

In this paper I will obtain the same result from another perspective: by representing these two distinct problems (Newtonian and Einstein's test body motion in central gravitational field) in a uniform way — as a geodesic motion. The difference in achieving "uniformity" is that whereas in [1] it was the same functional form of the equation in our case it will be the same geometric representation of the problem.

2 Relativistic epicycles from the geodesic deviation equation

Before going to the details let us start with some general comments. First of all, the problem of epicyclic frequencies has nothing to do with nonlinearity of Einstein's equations just because the Einstein's equations in general are dynamical equations for evolving the 3-geometry (see e.g. [5]). In the problem of epicyclic oscillations around circular orbits one has a kinematic problem of test bodies moving in static spacetime — the geometry is static and defined a priori. Hence the relevant question is how do the adjacent orbits of test particles behave.

The transition from Newtonian gravity to the Einstein's picture can be summarized in the following way. Newton's explanation why the planetary orbits are curved (circular, elliptical, parabolic or hyperbolic - for comets) was that it is the force of gravity from

the central body (the Sun) that makes them curved. In Newton's theory the nature of the force of gravity remained unexplained - it was taken for granted. Of course basic properties of the gravity force were explained e.g. the inverse square law, but not its nature. On the other hand, Einstein attempted at explaining the nature of gravity - there is no force field but the presence of massive central body makes the spacetime curved. The motion of test bodies takes place along geodesics; they are in a free motion but in a curved spacetime, that is why their trajectories are curved.

Therefore the (general relativistic) problem of epicyclic orbital oscillations in Schwarzschild spacetime is exactly the problem of geodesic deviation in Schwarzschild geometry. Stable circular orbits are stable in the sense that geodesic deviation equation solved along such circular orbit has oscillatory solutions.

One of the most recent papers presenting solution of the geodesic deviation equation for trajectories close to circular orbits in the Schwarzschild space-time is [6]. We will sketch main steps leading to the formula for radial epicylic frequency in Schwarzschild metric referring the interested reader to [6] for computational details.

In a pseudoriemannian manifold with the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1}$$

the curve $\gamma_s := x^\mu(s)$ parametrized by the affine parameter s is a geodesic if the tangent vector $u^\mu = \frac{dx^\mu}{ds}$ is paralelly transported along γ_s:

$$\frac{Du^\mu}{Ds} = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\sigma} u^\nu u^\sigma \tag{2}$$

Then consider close geodesic $\tilde{\gamma}_s$. The vector ξ^μ representing the separation between the geodesic γ_s and an adjacent geodesic $\tilde{\gamma}_s$ satisfies the geodesic deviation equation

$$\frac{D^2\xi^\mu}{Ds^2} = -R^\mu_{\nu\rho\sigma} u^\nu \xi^\rho u^\sigma \tag{3}$$

Now, let us take the Schwarzschild metric

$$ds^2 = (1 - \frac{2GM}{c^2 r}) c^2 dt^2 - \frac{1}{(1 - \frac{2GM}{c^2 r})} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \tag{4}$$

It is well known that circular orbits $r = R = const.$ are geodesics in the metric (4) and test particles move along such orbits with the angular velocity ω_K (Keplerian frequency) given by the formula [5]

$$\omega_K^2 = \frac{GM}{R^3} \tag{5}$$

If one considers a nearby geodesic (with respect to the circular one) and asks how does the separation between these two behave, the answer would come from solving the geodesic deviation equation (3). Technically one should express the components of the tangent vector $u^\mu = (u^t, u^r, u^\theta, u^\varphi)$ as well as the components of the Riemann tensor $R^\mu_{\nu\rho\sigma}$ (also the Christoffel symbols while calculating a covariant derivative, etc.) as evaluated along the circular orbit. Then one gets the system of four second-order differential equations. One of them — for ξ^θ component — reads $\frac{d^2\xi^\theta}{ds^2} = -\omega_K^2 \xi^\theta$ and is decoupled from the rest. The remaining three form the system of coupled linear second order differential equations

with constant coefficients (detailed calculations can be found in [6]) and the characteristic equation for this system (written in matrix form) reads:

$$\lambda^4(\lambda^2 + \frac{GM}{R^3}(1 - \frac{6GM}{c^2R})) = 0 \tag{6}$$

leading to the value of radial epicyclic oscillations with the frequency:

$$\omega_r^2 = \frac{GM}{R^3}(1 - \frac{6GM}{c^2R}) \tag{7}$$

In summary, the conclusion from solving the geodesic deviation equation in Schwarzschild spacetime is that behaviour of geodesics close to circular orbits can be represented as a superposition of (epicyclic) oscillations around circular orbit with two characteristic frequencies: the vertical epicyclic frequency – equal to Keplerian frequency ω_K of the reference orbit and radial epicyclic frequency ω_r.

3 Classical mechanics represented as problem of geodesics

It is well known that variational principles of classical mechanics make it possible to formulate the dynamics of Hamiltonian systems as geodesic flows on some Riemannian manifold. This picture comes quite naturally from the Maupertuis-Jacobi least action principle (motion of the system with fixed energy E between q' and q'' takes place along a path γ minimizing the Maupertuis-Jacobi action):

$$\delta S = \delta \int_{q'}^{q''} \sqrt{E - V(q)}\sqrt{a_{ij}\; dq^i dq^j} = 0 \tag{8}$$

and its formal resemblance to the variational formulation of geodesics in Riemannian geometry as curves extremalizing the distance. This is the simplest way to see desired correspondence and it is quoted in many textbooks on classical mechanics (e.g. in [7]). Below we give some steps along a straightforward "brute force" derivation of this result which could be instructive in seeing the role of time reparametrisation which is necessary in this picture.

Consider the classical mechanical system described by the Hamiltonian

$$H = H(p,q) = \frac{1}{2}a^{ij}p_ip_j + V(q) \tag{9}$$

The equations of motion for the q variables with respect to time parameter t (in Newtonian physics one has an absolute time) and corresponding to the Hamilton equations may be written as

$$\ddot{q}^j + \tilde{\Gamma}^j_{ks}\,\dot{q}^s\,\dot{q}^k = -a^{ji}\frac{\partial V(q)}{\partial q^i} \tag{10}$$

where $\tilde{\Gamma}^j_{ks}$ are the Christoffel symbols calculated with respect to a_{ij} metric and dots denote t - time derivative. Due to the force term this is, obviously, not a geodesic equation. It is simply the Newton's second law restated. The momentum variables are just linear combinations of velocities $p_i = a_{ij}\dot{q}^j$. Transformation to a geodesic motion (i.e. free motion in a curved space) is accomplished in two steps: (1) conformal transformation of

the metric a_{ij}, and (2) change of the time parameter along the orbit. More explicitly we equip the configuration space with the metric – the so called Jacobi metric

$$g_{ij} = 2(E - V(q))a_{ij} \tag{11}$$

(note that a_{ij} is read off from the kinetic energy term in the Hamiltonian and (in general) is allowed to vary as a function of the configuration space variable, $a_{ij} = a_{ij}(q)$)). Let us also call this Riemannian space i.e. configuration space accessible for the system and equipped with Jacobi metric — the Maupertuis-Jacobi manifold.
With respect to the metric (11) and the time parameter t it is not easy to see that the orbits are geodesics since there is a term appearing on the right hand side of the equation,

$$\frac{d^2}{dt^2}\, q^i + \Gamma^i_{\ jk}\, \frac{d}{dt} q^j\, \frac{d}{dt} q^k = -\frac{1}{E - V(q)}\frac{d}{dt} q^i \frac{\partial}{\partial q^k} V(q) \frac{d}{dt} q^k \tag{12}$$

where $\Gamma^i_{\ jk}$ now denote the Christoffel symbols associated with the Jacobi metric. However, if we reparametrize the orbit $q^i = q^i(s)$ in terms of the parameter s defined as

$$d\, s = 2\, (E - V)\, d\, t \tag{13}$$

the orbits will become affinely parametrized geodesics, i.e. the configuration space variables q^i satisfy the well known geodesic equation

$$\frac{d^2}{ds^2}\, q^i + \Gamma^i_{\ jk}\, \frac{d}{ds} q^j\, \frac{d}{ds} q^k = 0 \tag{14}$$

with no force term on the right hand side.

The information about the original force acting on the particle (as described by the potential $V(q)$ in the Hamiltonian (9)) has been encoded entirely in the definition of the Jacobi metric (11) and the definition of the new parameter s in (13) parametrizing the orbit.

Contemplating how nearby orbits behave (i.e. the local instability properties), it is natural to consider the geodesic deviation equation which describes the behavior of nearby geodesics (14).

This can be derived in a usual manner by subtracting the equations for the geodesics $q^i(s)$ and $q^i(s)+\xi^i(s)$ respectively or simply by disturbing the fiducial trajectory $(p_i(t), q^i(t))$,

$$\begin{aligned} \tilde{p}_i(t) &= p_i(t) + \eta_i(t), \\ \tilde{q}^i(t) &= q^i(t) + \xi^i(t) \end{aligned} \tag{15}$$

and substituting this directly into Hamilton's equations. In this way we also arrive momentarily, though tediously, at the geodesic deviation equation for the separation vector ξ,

$$\frac{D^2\, \xi^i}{D\, s^2} = -R^i_{jkl}\, u^j \xi^k u^l \tag{16}$$

Here $u^i = Dq^i/Ds$ is the tangent vector to the geodesic, ξ^j is the separation vector orthogonal to u. Note, that the covariant derivative D/Ds and Christoffel symbols are calculated with respect to the Jacobi metric (11).

4 Keplerian circular orbits represented as geodesics - epicyclic frequency in Newtonian's regime

Geometric formulation of the Kepler problem is very simple. The two body problem in Newtonian gravity is essentially two-dimensional. Therefore the Jacobi metric (in polar coordinates) reads:

$$ds^2 = 2(E - V(r))(dr^2 + r^2 d\varphi^2) \tag{17}$$

where: $V(r) = -\frac{GM}{r}$. Moreover all information carried by the Riemann curvature tensor is captured by the Gaussian curvature.

In this case the geodesic deviation equation (for an orthogonal Jacobi field ξ of geodesic deviation $g(\xi, u) = 0$) reads:

$$\frac{d^2}{ds^2}\xi^i + K_G \xi^i = 0 \tag{18}$$

where K_G is Gaussian curvature of respective Maupertuis-Jacobi manifold.

It is a simple exercise to calculate the Gaussian curvature of Maupertuis-Jacobi manifold for the Kepler problem. Let us denote $f(r)^2 := 2(E - V(r))$ so that Jacobi metric reads $ds^2 = f(r)^2(dr^2 + r^2 d\varphi^2)$, then let us consider the one-forms $\omega^1 := f(r)\, dr$ and $\omega^2 := f(r) r\, d\varphi$. Now, it is quite obvious that ω^1 is a closed form, and $d\omega^2 = -\frac{1}{f(r)^2 r}\frac{d(rf(r))}{dr}\omega^2 \wedge \omega^1$. Then from Cartan's equations one can easy read off the Gaussian curvature

$$K_G = -\frac{1}{f(r)^2 r}\frac{d}{dr}\left(\frac{1}{f(r)}\frac{d(rf(r))}{dr}\right) = -\frac{EGM}{4(rE + GM)^3} \tag{19}$$

Note that since the kinetic energy is positive definite $T := E - V(r) > 0$ the term $rE + GM$ is also positive and the sign of Gaussian curvature is determined by the sign of the energy. It is negative for $E > 0$ i.e. for hyperbolic orbits and positive for $E < 0$ i.e. for bound motion. The meaning of this results is that, in the first case, scattering of test particles on the center has sensitive dependence on initial conditions (problem is equivalent to congruence of geodesics on negatively curved manifold) while in the second case (equivalent to circular or elliptic orbits) the disturbed trajectories execute Keplerian epicyclic oscillations around the orbit of reference.

For the circular orbit of radius R (where $E = 1/2V(R)$) we have:

$$K_G = \frac{1}{GMR} \tag{20}$$

and (by virtue of geodesic deviation equation) $K_G = \omega_{0,J}^2$ where $\omega_{0,J}$ is the (Keplerian) epicyclic frequency in Jacobi geometry i.e. with respect to natural "time" s along the geodesic.

Recalling that $ds = 2(E - V(R))dt = \frac{GM}{R}dt$ one can easily recover Keplerian epicyclic frequency in Newtonian picture i.e. with respect to the Newtonian time t

$$\omega_0^2 = \frac{GM}{R^3} = \omega_K^2 \tag{21}$$

and it turns out to be equal to the orbital (Keplerian) frequency ω_K.

Let us also remark on the equation (20). The quantity in the denominator is in fact equal to the Keplerian angular momentum (per unit test body mass) squared. In our

approach it turns out that for circular orbits the specific angular momentum squared is equal to the inverse Gaussian curvature of the Jacobi manifold in which the motion takes place along geodesics. In the picture developed by Abramowicz and Kluźniak the specific angular momentum had an interpretation of the geometric mean of the gravitational radius of central body and the radius of curvature of particle's orbit.

5 Conclusion

Conclusion of this note is that by applying a uniform representation of the problem i.e. by representing physical trajectories as a problem of geodesics in some manifold one can see the geometric origin of the difference in epicyclic frequencies (describing the behaviour of trajectories adjacent to circular orbits) calculated in Newtonian and General Relativistic regimes. This result is in agreement with that of [1] although it has been derived in a different manner. One may say that instead of applying the Feynman's principle "the same equations have the same solutions" we have sucessfully applied the principle of "comparing comparable things" by working in the same geometric representation of the problem.

References

[1] Abramowicz M.A. and Kluźniak W., Gen. Rel. Grav. **35**, 69–77, 2003

[2] Abramowicz M.A., Almergren G.J.E., Kluźniak W., Thampan A.V., Wallinder F., Class.Quant.Grav. **19**, L52-L62, 2002
Abramowicz M.A., Kluźniak W., Astron. Astrophys., **374**, L19-L20, 2001
Kluźniak W., Abramowicz M.A., Acta Phys. Polon. B **32**, 3605-3612, 2001

[3] Strohmayer T.E., ApJ **552**, L49-L53, 2001

[4] Abramowicz M.A., Carter B., Lasota J.-P., Gen. Rel. Grav. **20**, 1173, 1988

[5] Misner C.W., Thorne K.S., Wheeler J.A., *Gravitation*, Freeman, New York, 1973

[6] Kerner R., Van Holten J.W., Colistete R.,Jr., *Class. Quant. Grav.*, **18**, 4725-4742, 2001

[7] Arnold V.I., *Mathematical Methods of Classical Mechanics*, Springer Verlag, 1978